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EXTENDED RECTIFYING CURVES AS NEW KIND OF MODIFIED DARBOUX VECTORS

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Abstract. Rectifying curves are defined as curves whose position vectors always lie in rectifying plane. The centrode of a unit speed curve in $\mathbb{E}^3$ with nonzero constant curvature and non-constant torsion (or nonzero constant torsion and non-constant curvature) is a rectifying curve. In this paper, we give some relations between non-helical extended rectifying curves and their Darboux vector fields using any orthonormal frame along the curves. Furthermore, we give some special types of ruled surface. These surfaces are formed by choosing the base curve as one of the integral curves of Frenet vector fields and the director curve δ as the extended modified Darboux vector fields.

Keywords: rectifying curve, centrodes, Darboux vector, conical geodesic curvature.

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1. Introduction

From elementary differential geometry it is well known that at each point of a curve α, its planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are known as the osculating plane, the rectifying plane and the normal plane, respectively. A curve called twisted curve has non-zero curvature functions in the Euclidean 3-space. Rectifying curves are introduced by B. Y. Chen in [3] as space curves whose position vector always lies in its rectifying plane, spanned by the tangent and the binormal vector fields T and B of the curve. Accordingly, the position vector with respect to some chosen origin, of a rectifying curve α in $\mathbb{E}^3$, satisfies the equation

$$\alpha(s) = \lambda(s)T(s) + \mu(s)B(s)$$

for some functions $\lambda(s)$ and $\mu(s)$. He proved that a *twisted curve* is congruent to a rectifying curve if and only if the ratio $\dfrac{\tau}{\varkappa}$ is a non-constant linear function of arclength s. Subsequently Ilarslan and Nesovic generalized the rectifying curves in Euclidean 3-space to Euclidean 4-space [10].

A necessary and sufficient condition for the curve to be a general helix is that the ratio of curvature to torsion is constant, i.e. the harmonic curvature function $H = \dfrac{\tau}{\kappa}$ of the curve is constant. In a special case, if both of $\varkappa$ and τ are non-zero constants, then the curve is called a circular helix. It is known that straight line and circle are degenerate-helix examples.

Then, Izumiya and Takeuchi [11] defined slant helices and conical geodesic curves in Euclidean $3-$space $\mathbb{E}^3$. A slant helix is a curve whose principal normal makes a constant angle with a fixed

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direction and a curve with non-zero curvature called *conical geodesic curve* if the function H' is a constant function. Then, they proved that α is a slant helix if and only if

$$\sigma = \frac{H'}{\kappa(1+H^2)^{3/2}} = const.$$

Also, they gave a classification of special developable surfaces under the condition of the existence of a slant curve and conical geodesic curve as a geodesic.

Then, rectifying curves as centrodes and extremal curves were introduced by Chen and Dillen in [4]. They showed that the centrode of a unit speed curve in $\mathbb{E}^3$ with nonzero constant curvature $\varkappa$ (nonconstant curvature) and nonconstant torsion τ (nonzero constant torsion) is a rectifying curve. This characterization is only for rectifying curves which have a non zero constant curvature and non-constant torsion (or a non zero constant torsion and non constant curvature). Recently, Chen [5] has surveyed six research topics in differential geometry in which the position vector field plays important roles. In this survey article the author has explained the relationship between position vector fields and mechanics, dynamics. Furthermore, in another paper the same author [6, 7] has introduced and studied the notion of rectifying submanifolds in Euclidean spaces. He has also proved that a Euclidean submanifold is rectifying if and only if the tangential component of its position vector field is a concurrent vector field. Moreover, rectifying submanifolds with arbitrary codimension have been completely determined. Another paper about rectifying curves is [16]. Yılmaz et al. considered nonhelical rectifying curves using an orthonormal moving frame in Minkowski 3-space. Then, they gave some relations between nonhelical rectifying curves and their Darboux vectors, and proved that modified Darboux vectors of curves are rectifying curves. In [8], Chen has studied geodesics on arbitrary cone $\mathbb{E}^3$ and showed that a curve on a cone in $\mathbb{E}^3$ is a geodesic if and only if it is a rectifying curve or an open portion of a ruling. In [9], Deshmukh et al. have studied rectifying curves via the dilation of unit speed curves on the unit sphere S^2 in the Euclidean space $\mathbb{E}^3$. They have also proved that if a unit speed curve $\alpha(s)$ in $\mathbb{E}^3$ is neither a planar curve nor a helix, then its dilated centrode $\beta(s) = \rho(s)d(s)$, with dilation factor ρ, is always a rectifying curve, where ρ is the radius of curvature of α.

Scofield [13] has defined a curve of constant precession. The Darboux vector field of the curve revolves about a fixed axis with constant angle and constant speed.

In [12], Kula and Yaylı have investigated spherical tangent indicatrix and binormal indicatrix of a slant helix. They have obtained that the spherical images are spherical helices. Moreover, they have shown that a curve of constant precession is a slant helix.

Ali [1] defined a curve called k-slant helix and gave various characterizations. After this study, Uzunoğlu et.al [15] defined $C-$slant curves with respect to the alternative frame $\{N, C, W, f, g\}$ (also called slant-slant helix in [1] and clad helix in [14]) and curves of $C-$constant precession. They showed that a unit-speed curve α is a $C-$slant helix if and only if

$$\Gamma = \frac{\sigma'}{f(1+\sigma^2)^{3/2}} = const,$$

where $\sigma = g/f$. Moreover, they obtained that the tangent image of $C-$slant curves are spherical slant curves. Also, they showed that a curve of $C-$constant precession is a $C-$slant helix.

In this generalization, we give some relations between non-helical extended rectifying curves and their Darboux vector fields using any orthonormal frame along the curves. Moreover, we obtain some relations between the papers [4] and [11].

2. Basic notions and arguments

Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be an arbitrary curve in three dimensional Euclidean space. Recall that the curve is said to be a unit speed curve (or parameterized by arclength functions) if $\langle \alpha'(s), \alpha'(s) \rangle = 1$ here $\langle ., . \rangle$ denotes the standard inner product of $\mathbb{E}^3$ given by $\langle X, Y \rangle = \sum_{i=1}^{3} x_i y_i$ for each $X = (x_1, x_2, x_3)$ and $Y = (y_1, y_2, y_3) \in \mathbb{R}^3$. In particular, the norm of a vector $X \in \mathbb{R}^3$ is given by $\|X\| = \sqrt{\langle X, X \rangle}$. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve which has at least four continuous derivatives.

The notion of curvature of the smooth curve α which is invariant under the motions is important for its shape. The curvature of the curve at a point is a measure of how sensitive its tangent line is to moving the point to other nearby points. The curvature of a plane curve is a quantity which measures the amount by which the curve differs from being a straight line. It measures the rate at which the direction of a tangent to the curve changes. The curvature of a straight line is identically zero and the curvature of a circle whose radius R is defined by $\varkappa = \dfrac{1}{R}$.

A plane curve is defined by its curvature function $\varkappa$ which is a function of arclength parameter s. On the other hand, a space curve is completely determined by the curvature $\varkappa$ which is the magnitude of the acceleration of a particle moving with unit speed along a curve and the torsion τ which measures how sharply it is twisting out of the plane of curvature.

A moving frame of a curve α with arclength parameter s in the Euclidean space $\mathbb{E}^3$ in more detail is defined to be a 4-tuple of vectors drawn from $\mathbb{E}^3$, $(\alpha(s), N_1, N_2, N_3)$; where $\alpha(s)$ is a choice on the curve α and (N_1, N_2, N_3) is an orthonormal basis of the vector space $\mathbb{E}^3$ based on $\alpha(t)$. In mathematics, a moving frame is a flexible generalization of the notion of an ordered basis of a vector space often used to study the extrinsic differential geometry of smooth manifolds embedded in a homogeneous space. Here, $N_2(s)$ is a unit normal vector and $N_3(s)$ is perpendicular to the vectors $N_1(s)$ and $N_2(s)$, that is, $N_3(s) = N_1(s) \times N_2(s)$ for every parameter s.

Since the frame (N_1, N_2, N_3) is orthonormal under the inner product of Euclidean space $\mathbb{E}^3$, change of the frame and its derivative is given by the following matrix

$$\begin{bmatrix} N_1' \\ N_2' \\ N_3' \end{bmatrix} = \begin{bmatrix} 0 & \varkappa_1 & \varkappa_2 \\ -\varkappa_1 & 0 & \varkappa_3 \\ -\varkappa_2 & -\varkappa_3 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ N_3 \end{bmatrix} \tag{1}$$

where the exterior derivative of $N_1(s)$ with respect to s decomposes uniquely as $N_1'(s) = \sum_{j=1}^{2} \varkappa_{j+1} N_{j+1}$ which is the curvature vector of α, while $\varkappa_3$ measures the twisting of this framing. The Frenet-Serret frame, Bishop frame and Darboux frame on a curve are some simple examples of a moving frame. Now, let us see this fact using the frame given Eq.(1).

Assume that the curvature vector of the curve α never vanishes then we can write that $\varkappa_2$ is zero and $\varkappa = \varkappa_1 = \|N_1'(s)\|$. In this case, $N_2 = N$ is principal normal vector of the Serret-Frenet frame $(T, N, B = T \times N)$, whose twisting out of the plane of curvature $\varkappa_3$ is the torsion which measures how sharply it is twisting out of the plane of curvature.

If the Eq. (1) is defined by the condition $\varkappa_3 = 0$ then we can easily obtain the *parallel transport frame* or *Bishop frame*. Bishop [2] defined a new frame for a curve and he called it Bishop frame which is well defined even if the curve has vanishing second derivative in $3-$dimensional Euclidean space. Using the equations $N_1(s) = N(s)$, $N_2(s) = C(s)$ and $N_3(s) = W(s)$ we get the alternative moving frame defined by [15].

Let M be a surface with a unit normal vector η and α be an arclenghted curve on M. If we choose $\eta = N_2$ and $\xi = N_3$ defined by $\alpha' \times \eta = \xi$ (ξ is called a unit normal vector field of M)

then $(\alpha' = T, \eta, \xi)$ is called *Darboux frame.* Here, the derivative of the curvature vector of the curve α as $T' = \varkappa_g \eta + \varkappa_n \xi$ where $\varkappa_g = \varkappa_1$ is the geodesic curvature of α on M and $\varkappa_n = \varkappa_2$ is the normal curvature of the surface M in the direction T. It is well known that, if $\varkappa_g = 0$, then the curve is called a *geodesic*, if $\varkappa_n = 0$, then the curve is called an *asymptotic* and if $\tau_r = \varkappa_3 = 0$, then the curve is called a *principal curve.*

In the theory of space curves, the Darboux vector is the areal velocity vector of the moving frame of a space curve. It is also called angular momentum vector, because it is directly proportional to angular momentum. The direction of the Darboux vector is that of the instantaneous axis of rotation, its angular speed is $\vartheta_{ang} = \sqrt{\varkappa_1^2 + \varkappa_2^2 + \varkappa_3^2}$. In terms of the moving frame apparatus, the general Darboux vector field $\mathcal{D}$ can be expressed as

$$\mathcal{D}(s) = \varkappa_3(s) N_1(s) - \varkappa_2(s) N_2(s) + \varkappa_1(s) N_3(s) \tag{2}$$

and it has the following symmetrical properties:

$$\begin{aligned} \mathcal{D} \times N_1 &= N_1', \\ \mathcal{D} \times N_2 &= N_2', \\ \mathcal{D} \times N_3 &= N_3', \end{aligned}$$

where $\times$ is the wedge product in Euclidean space $\mathbb{E}^3$. Considering Serret Frenet apparatus $\{T, N, B, \varkappa, \tau\}$ of any unit speed curve α, Izumiya and Takeuchi [11] defined the modified Darboux vector field $\widetilde{D} = \left(\frac{\tau}{\varkappa}\right)(s)T(s) + B(s)$ under the condition that $\varkappa(s) \neq 0$ and the unit modified Darboux vector field

$$\widetilde{\widetilde{D}} = \left(\frac{1}{\sqrt{\tau^2 + \varkappa^2}}\right)(s)\left\{\tau(s)T(s) + \varkappa(s)B(s)\right\}$$

along the curve α. Also, we can construct a new modified Darboux vector field. $\overline{D} = T(s) + \left(\frac{\varkappa}{\tau}\right)(s)B(s)$ under the condition that $\tau(s) \neq 0$.

3. Extended rectifying curves as a new kind of modified Darboux vectors

In this section, we define some modified Darboux vectors which are special cases of the general Darboux vector field in Eq.(2-2). First of all we give some well known useful notions about rectifying curves and then we extend rectifying curves and obtain some characterizations of them. Here one important point is that the orthonormal frame by the condition $\varkappa_2 = 0$ is different from the Frenet frame. Because only if $\varkappa_1 = \varkappa$ and $\varkappa_3 = \tau$ in Eq.(1) then the orthonormal frame is the Frenet frame.

After this, we will consider the orthonormal frame with the condition zero curvature $\varkappa_2$ but we note that the frame is Frenet frame for the above special case.

Theorem 3.1. [3] *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with $\varkappa \neq 0$. Then, α is congruent to a rectifying curve if and only if the ratio of torsion and curvature of the curve is a nonconstant linear function in arclength function s, i.e., $\frac{\tau}{\varkappa} = c_1 s + c_2$ for some constants c_1 and c_2 with $c_1 \neq 0$.*

Definition 3.1. *For a regular curve α in $\mathbb{E}^3$ with $\varkappa \neq 0$, the curve given by the Darboux vector $d = \tau T + \varkappa B$ is called the* centrode *of α and the curves $C_{\pm} = \alpha \pm d$ are called the* co-centrodes *of α.*

Theorem 3.2. [4] *The centrode of a unit speed curve in $\mathbb{E}^3$ with nonzero constant curvature $\varkappa$ (nonconstant curvature) and nonconstant torsion τ (nonzero constant torsion) is a rectifying curve.*

Conversely, every rectifying curve in $\mathbb{E}^3$ is the centrode of some unit speed curve with nonzero constant curvature and nonconstant torsion (nonzero constant torsion and nonconstant curvature).

Definition 3.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with orthonormal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$. We define a Darboux vector field $\mathbf{D} = \varkappa_3(s)N_1(s) + \varkappa_1(s)N_3(s)$ along the curve α. Also, we define the vector fields called modified Darboux vector fields $\widetilde{\mathbf{D}} = \left(\frac{\varkappa_3}{\varkappa_1}\right)(s)N_1(s) + N_3(s)$ under the condition $\varkappa_1(s) \neq 0$ and $\overline{\mathbf{D}} = N_1(s) + \left(\frac{\varkappa_1}{\varkappa_3}\right)(s)N_3(s)$ under the condition $\varkappa_3(s) \neq 0$.*

Definition 3.3. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with the frame apparatus $\left\{N, C = \frac{N'}{\|N'\|}, W = N \times C, f, g\right\}$ defined by Uzunoğlu et al* [15]. *We define a Darboux vector field as $\mathbb{D} = g(s)N(s) + f(s)W(s)$ along the curve α. Also, we define the vector fields called modified Darboux vector fields $\widetilde{\mathbb{D}} = \left(\frac{g}{f}\right)(s)N(s) + W(s)$ under the condition $f(s) \neq 0$ and $\overline{\mathbb{D}} = N(s) + \left(\frac{f}{g}\right)(s)W(s)$ under the condition $g(s) \neq 0$.*

Theorem 3.3. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with orthonormal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$. Then the following statements hold.*

- **i:** $\left(\frac{\varkappa_3}{\varkappa_1}\right)(s)$ is the geodesic curvature of the curve $\beta = N_1(s)$ where s_β is arc-length parameter of the curve β and $\varkappa_1(s) \neq 0$.
- **ii:** $\left(\frac{\varkappa_1}{\varkappa_3}\right)(s)$ is the geodesic curvature of the curve $\gamma = N_3(s)$ where s_γ is arc-length parameter of the curve γ and $\varkappa_3(s) \neq 0$.

Proof. For the curve $N_1(s)$ with its Frenet apparatus $\{T_\beta, N_\beta, B_\beta, \varkappa_\beta, \tau_\beta\}$ we have

$$T_\beta(s_\beta)\frac{ds_\beta}{ds} = \varkappa_1(s)N_2(s) \tag{3}$$

From (3), we get $\frac{ds_\beta}{ds} = \varkappa_1(s)$ and $T_\beta(s_\beta) = N_2(s)$. Differentiating the last equality with respect to s and by using (1), we get

$$\begin{aligned} \varkappa_\beta N_\beta(s_\beta) &= \{-\varkappa_1(s)N_1(s) + \varkappa_3(s)N_3(s)\}\frac{ds}{ds_\beta}, \\ &= -N_1(s) + \left(\frac{\varkappa_3}{\varkappa_1}\right)(s)N_3(s), \end{aligned} \tag{4}$$

If we take the norm of the last equality, we obtain

$$\varkappa_\beta = \sqrt{1 + \left(\frac{\varkappa_3}{\varkappa_1}\right)^2(s)}. \tag{5}$$

On the other hand, since $k_n = 1$ and $\varkappa_\beta^2 = (k_n)^2 + (k_g)^2$, we can easily see that $(k_g)_\beta = \frac{\varkappa_3}{\varkappa_1}$ for $\varkappa_1(s) \neq 0$.

Similarly, for the curve $\gamma(s_\gamma) = N_3(s)$ with its Frenet vectors $\{T_\gamma, N_\gamma, B_\gamma, \varkappa_\gamma, \tau_\gamma\}$, we can easily see that $(k_g)_\gamma = \frac{\varkappa_1}{\varkappa_3}$ for $\varkappa_3(s) \neq 0$. This completes the proof. $\square$

The following two corallaries are given without their proofs. Because they can proved with the similar method of the above proof of theorem.

Corollary 3.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with Frenet frame apparatus $\{T, N, B, \varkappa, \tau\}$. Then the following statements hold.*

- **i:** $\left(\dfrac{\tau}{\varkappa}\right)(s)$ is the geodesic curvature of the curve $\beta = T(s)$ where s_β is arc-length parameter of the curve β and $\varkappa(s) \neq 0$.
- **ii:** $\left(\dfrac{\varkappa}{\tau}\right)(s)$ is the geodesic curvature of the curve $\gamma = B(s)$ where s_γ is arc-length parameter of the curve γ and $\tau(s) \neq 0$.

Corollary 3.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with the frame apparatus $\left\{N, C = \dfrac{N'}{\|N'\|}, W = N \times C, f, g\right\}$ defined by Uzunoğlu et al.* [15]. *Then the following statements hold.*

- **i:** $\left(\dfrac{g}{f}\right)(s)$ is the geodesic curvature of the curve $\beta = N(s)$ where s_β is arc-length parameter of the curve β and $f(s) \neq 0$.
- **ii:** $\left(\dfrac{f}{g}\right)(s)$ is the geodesic curvature of the curve $\gamma = W(s)$ where s_γ is arc-length parameter of the curve γ and $g(s) \neq 0$.

Definition 3.4. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with orthonormal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$. Then, the curve $\bar{\beta}(s_\beta) = \int N_1(s)ds$ is called* conical geodesic curve *if $\left(\dfrac{\varkappa_3}{\varkappa_1}\right)'(s)$ is a non-zero constant function where $\varkappa_1(s) \neq 0$. Similarly, the curve $\bar{\gamma}(s_\gamma) = \int N_3(s)ds$ is called another* conical geodesic curve *if $\left(\dfrac{\varkappa_1}{\varkappa_3}\right)'(s)$ is a non-zero constant function where $\varkappa_3(s) \neq 0$.*

In the sequel, we will consider the curves $\bar{\beta}$ and $\bar{\gamma}$ are $\int N_1(s)ds$ and $\int N_3(s)ds$, respectively.

Theorem 3.4. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with any orthonormal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$. Then, the modified Darboux vector field $\widetilde{\mathbf{D}}$ with non-constant geodesic curvature $(k_g)_\beta$ of the curve β is rectifying curve if and only if every rectifying curve is the modified Darboux vector of some unit speed curve with non-constant geodesic curvature. Similarly, the modified Darboux vector field $\overline{\mathbf{D}}$ with non-constant geodesic curvature $(k_g)_\gamma$ of the curve γ is rectifying curve if and only if every rectifying curve is the modified Darboux vector of some unit speed curve with non-constant geodesic curvature.*

Proof. Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with orthonormal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$. Then, the modified Darboux vector field $\widetilde{\mathbf{D}} = \left(\dfrac{\varkappa_3}{\varkappa_1}\right)(s)N_1(s) + N_3(s)$ under the condition that $(k_g)_\beta$ is non-constant. Differentiating $\widetilde{\mathbf{D}}$ with respect to s and applying (1), we have

$$\frac{d\widetilde{\mathbf{D}}}{d\widetilde{s}}\frac{d\widetilde{s}}{ds} = \left(\frac{\varkappa_3}{\varkappa_1}\right)'(s)N_1(s)$$

which implies that as the unit tangent vector field of $\widetilde{\mathbf{D}}$ denoted by $\mathcal{T}_{\widetilde{\mathbf{D}}}$ is parallel to the vector field $T(s)$, that is,

$$\mathcal{T}_{\widetilde{\mathbf{D}}} = N_1(s). \tag{6}$$

On the other hand, we can easily see that

$$\widetilde{s} = \left(\frac{\varkappa_3}{\varkappa_1}\right)(s) + \lambda, \lambda \in \mathbb{R}. \tag{7}$$

Differentiating the Eq.(6) with respect to s, we get

$$\varkappa_{\widetilde{\mathbf{D}}} \mathcal{N}_{\widetilde{\mathbf{D}}}(\widetilde{s}) \left(\frac{\varkappa_3}{\varkappa_1}\right)'(s) = \varkappa_1(s) N_2(s).$$

So, the principal vector field of $\widetilde{\mathbf{D}}$ denoted by $\mathcal{N}_{\widetilde{\mathbf{D}}}$ is parallel to the vector field $N_2(s)$, that is,

$$\mathcal{N}_{\widetilde{\mathbf{D}}} = N_2(s) \tag{8}$$

and the curvature of $\widetilde{\mathbf{D}}$ is given by

$$\varkappa_{\widetilde{\mathbf{D}}} = \frac{\varkappa_1}{\left(\frac{\varkappa_3}{\varkappa_1}\right)'}. \tag{9}$$

If we differentiate the Eq.(7) with respect to s, then we can easily obtain that

$$\mathcal{B}_{\widetilde{\mathbf{D}}} = N_3(s) \tag{10}$$

and the torsion of $\widetilde{\mathbf{D}}$ is given by

$$\tau_{\widetilde{\mathbf{D}}} = \frac{\varkappa_3}{\left(\frac{\varkappa_3}{\varkappa_1}\right)'} \tag{11}$$

The equations (9) and (10) give us

$$\frac{\tau_{\widetilde{\mathbf{D}}}}{\varkappa_{\widetilde{\mathbf{D}}}} = \left(\frac{\varkappa_3}{\varkappa_1}\right)(s) = \widetilde{s} - \lambda, \lambda \in \mathbb{R} \tag{12}$$

and using the Theorem (3), we have the modified Darboux vector field $\widetilde{\mathbf{D}}$ with non-constant geodesic curvatures $(k_g)_\beta$ of the curve β is rectifying curve,

Conversely, we assume that $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed rectifying curve in Euclidean space $\mathbb{E}^3$. Then, using a similar method of the proof of Theorem (1) in [4] we obtain that the modified Darboux vector field $\widetilde{\mathbf{D}}$ is rectifying curve.

Similarly, we can easily proved this fact for the modified Darboux vector field $\overline{\mathbf{D}} = N_1(s) + \left(\frac{\varkappa_1}{\varkappa_3}\right)(s) N_3(s)$ under the condition $\varkappa_3(s) \neq 0$.

These complete the proof. □

Corollary 3.3. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed* ***nonhelical curve*** *in Euclidean space $\mathbb{E}^3$ with Frenet frame apparatus $\{T, N, B, \varkappa, \tau\}$. Then, the modified Darboux vector field $\widetilde{D}$ is rectifying curve if and only if every rectifying curve is the modified Darboux vector of some unit speed nonhelical curve. Similarly, the modified Darboux vector field $\overline{D}$ is rectifying curve if and only if every rectifying curve is the modified Darboux vector of some unit speed nonhelical curve.*

Proof. It is obvious from the above theorem using Serret-Frenet formulas. □

Corollary 3.4. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed* ***nonslant curve*** *in Euclidean space $\mathbb{E}^3$ with the frame apparatus $\left\{N, C = \frac{N'}{\|N'\|}, W = N \times C, f, g\right\}$ defined by Uzunoğlu et al [15]. Then, the modified Darboux vector field $\widetilde{\mathbb{D}}$ is rectifying curve if and only if every rectifying curve is the modified Darboux vector of some unit speed nonslant curve. Similarly, the modified Darboux vector field $\overline{\mathbb{D}}$ is rectifying curve if and only if every rectifying curve is the modified Darboux vector of some unit speed non W-slant curve.*

Proof. It is obvious from the above theorem using alternative frame formulas in [15]. □

Theorem 3.5. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with orthonormal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$. Then, the curves $\bar{\beta}$ and $\bar{\gamma}$ are rectifying curves if and only if the curves $\bar{\beta}$ and $\bar{\gamma}$ are conical geodesic curves.*

Proof. Since the curve $\bar{\beta}$ is rectifying curve $\frac{\tau_{\bar{\beta}}}{\varkappa_{\bar{\beta}}}$ is a linear function, that is, $\frac{\tau_{\bar{\beta}}}{\varkappa_{\bar{\beta}}} = \lambda s + \mu$, λ and μ are constants. By using the equations (3.3) and (3.4) we get

$$\frac{\tau_{\bar{\beta}}}{\varkappa_{\bar{\beta}}} = \frac{\varkappa_3}{\varkappa_1} = \lambda s + \mu, \lambda \in \mathbb{R} - \{0\} \text{ and } \mu \in \mathbb{R}.$$

Then $\left(\frac{\varkappa_3}{\varkappa_1}\right)'$ is a nonzero constant function. So, the curve $\bar{\beta}$ is conical geodesic curve.

Conversely, assume that $\bar{\beta}$ is conical geodesic curve. Then $\left(\frac{\varkappa_3}{\varkappa_1}\right)'$ is a nonzero constant function. Using the relations between the curvatures of $\bar{\beta}$ and the curvatures of α, we have $\frac{\varkappa_3}{\varkappa_1} = \frac{\tau_{\bar{\beta}}}{\varkappa_{\bar{\beta}}} = \lambda s + \mu$, $\lambda \in \mathbb{R} - \{0\}$ and $\mu \in \mathbb{R}$. Consequently, the curve $\bar{\beta}$ is rectifying curve.

Similarly, we can easily proved this fact for the curve $\bar{\gamma}$.

These complete the proof. □

The following two corollaries are given without their proofs. Because they can proved with the similar method of the above proof of theorem.

Corollary 3.5. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with Frenet frame apparatus $\{T, N, B, \varkappa, \tau\}$. Then, the curves $\int T(s)ds$ and $\int B(s)ds$ are rectifying curves if and only if the ratios $\left(\frac{\tau}{\varkappa}\right)'$ and $\left(\frac{\varkappa}{\tau}\right)'$ are non-zero constant functions, respectively.*

Corollary 3.6. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with the frame apparatus $\left\{N, C = \frac{N'}{\|N'\|}, W = N \times C, f, g\right\}$ defined by Uzunoğlu et al*[15]. *Then, the curves $\int N(s)ds$ and $\int W(s)ds$ are rectifying curves if and only if the ratios $\left(\frac{g}{f}\right)'$ and $\left(\frac{f}{g}\right)'$ are non-zero constant functions, respectively.*

Theorem 3.6. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with any orthonormal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$. Then, the curve $\int N_1(s)ds$ is rectifying curve with non-constant ratio $\frac{\varkappa_3}{\varkappa_1}$ and $\varkappa_1 \neq 0$ if and only if one of the curves $\widetilde{\mathbf{C}}_{\pm} = \int N_1(s)ds \pm \widetilde{\mathbf{D}}(s)$ is a rectifying curve.*

Similarly, the curve $\int N_3(s)ds$ is rectifying curve with non-constant ratio $\frac{\varkappa_1}{\varkappa_3}$ and $\varkappa_3 \neq 0$ if and only if one of the curves $\overline{\mathbf{C}}_{\pm} = \int N_3(s)ds \pm \overline{\mathbf{D}}(s)$ is a rectifying curve.

Corollary 3.7. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed* ***nonhelical curve*** *in Euclidean space $\mathbb{E}^3$ with Frenet frame apparatus $\{T, N, B, \varkappa, \tau\}$. Then, the curve α is rectifying curve with non-constant ratio $\frac{\tau}{\varkappa}$ and $\varkappa \neq 0$ if and only if one of the curves $\widetilde{\mathbf{C}}_{\pm} = \int T(s)ds \pm \widetilde{D}(s)$ is a rectifying curve.*

Similarly, the curve α is rectifying curve with non-constant ratio $\frac{\varkappa}{\tau}$ and $\tau \neq 0$ if and only if one of the curves $\overline{\mathbf{C}}_{\pm} = \int B(s)ds \pm \overline{D}(s)$ is a rectifying curve.

Corollary 3.8. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed* ***nonslant curve*** *in Euclidean space $\mathbb{E}^3$ with the frame apparatus $\left\{N, C = \dfrac{N'}{\|N'\|}, W = N \times C, f, g\right\}$ defined by Uzunoğlu et al*[15]*. Then, the curve $\int N(s)ds$ is rectifying curve with non-constant ratio $\dfrac{g}{f}$ and $f \neq 0$ if and only if one of the curves $\widetilde{\mathbf{C}}_{\pm} = \int N(s)ds \pm \widetilde{\mathbb{D}}(s)$ is a rectifying curve.*

Similarly, the curve $\int W(s)ds$ is rectifying curve with non-constant ratio $\dfrac{f}{g}$ and $g \neq 0$ if and only if one of the curves $\overline{\mathbf{C}}_{\pm} = \int W(s)ds \pm \overline{\mathbb{D}}(s)$ is a rectifying curve.

Theorem 3.7. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with any orthonormal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$. If the ratio $\dfrac{\varkappa_3}{\varkappa_1} = \tan s$ with $\varkappa_1 \neq 0$ then $\widetilde{\mathbf{D}}(s) = \sec s\mathbf{Y}(s)$ is a rectifying curve where $\mathbf{Y}(s) = \sin sN_1(s) + \cos sN_3(s)$ is a curve in S^2.*

Similarly, if the ratio $\dfrac{\varkappa_1}{\varkappa_3} = \tan s$ with $\varkappa_3 \neq 0$ then $\overline{\mathbf{D}}(s) = \sec s\mathbf{Y}(s)$ is a rectifying curve where $\mathbf{Y}(s) = \cos sN_1(s) + \sin sN_3(s)$ is a curve in S^2.

Proof. From the Definition (3.1) we have $\widetilde{\mathbf{D}} = \left(\dfrac{\varkappa_3}{\varkappa_1}\right)(s)N_1(s) + N_3(s)$ or $\widetilde{\mathbf{D}} = \tan sN_1(s) + N_3(s) = \sec s(\sin sN_1(s) + \cos sN_3(s))$. Similarly, we can easily prove the other case. These complete the proof. □

Corollary 3.9. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed* ***nonhelical curve*** *in Euclidean space $\mathbb{E}^3$ with Frenet frame apparatus $\{T, N, B, \varkappa, \tau\}$. If the ratio $\dfrac{\tau}{\varkappa} = \tan s$ with $\varkappa \neq 0$ then $\widetilde{D}(s) = \sec sY(s)$ is a rectifying curve where $Y(s) = \sin sT(s) + \cos sB(s)$ is a curve in S^2.*

Similarly, if the ratio $\dfrac{\varkappa}{\tau} = \tan s$ with $\tau \neq 0$ then $\overline{D}(s) = \sec sY(s)$ is a rectifying curve where $Y(s) = \cos sT(s) + \sin sB(s)$ is a curve in S^2.

Corollary 3.10. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed* ***nonslant curve*** *in Euclidean space $\mathbb{E}^3$ with the frame apparatus $\left\{N, C = \dfrac{N'}{\|N'\|}, W = N \times C, f, g\right\}$ defined by Uzunoğlu et al* [15]*. If the ratio $\dfrac{g}{f} = \tan s$ with $f \neq 0$ then $\widetilde{\mathbb{D}}(s) = \sec s\mathbb{Y}(s)$ is a rectifying curve where $\mathbb{Y}(s) = \sin sN(s) + \cos sW(s)$ is a curve in S^2.*

Similarly, if the ratio $\dfrac{f}{g} = \tan s$ with $g \neq 0$ then $\overline{\mathbb{D}}(s) = \sec s\mathbb{Y}(s)$ is a rectifying curve where $\mathbb{Y}(s) = \cos sN(s) + \sin sW(s)$ is a curve in S^2.

Example 3.1. *We draw the picture of the curve*

$$\alpha(s) = \begin{pmatrix} \dfrac{3-2\sqrt{2}}{2\sqrt{2}}\sin((\sqrt{2}+1)s) - \dfrac{3+2\sqrt{2}}{2\sqrt{2}}\sin((\sqrt{2}-1)s), \\ -\dfrac{3-2\sqrt{2}}{2\sqrt{2}}\cos((\sqrt{2}+1)s) + \dfrac{3+2\sqrt{2}}{2\sqrt{2}}\cos((\sqrt{2}-1)s), \\ \dfrac{1}{\sqrt{2}}\sin s \end{pmatrix}$$

in figure 1. Then we illustrate its modified centrode $\overline{D}(s)$ and the spherical curve $Y(s)$ regarded $\overline{D}(s)$ in figure 2.

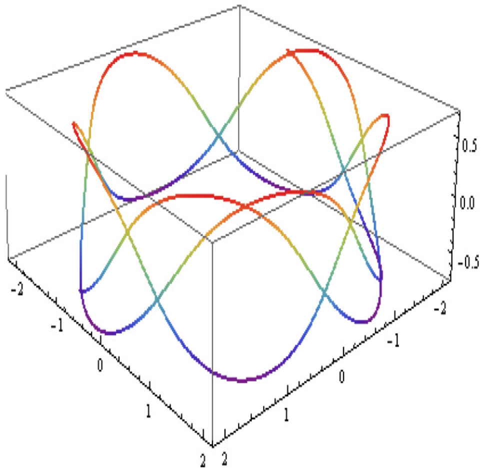

Figure 1. The curve $\alpha(s)$.

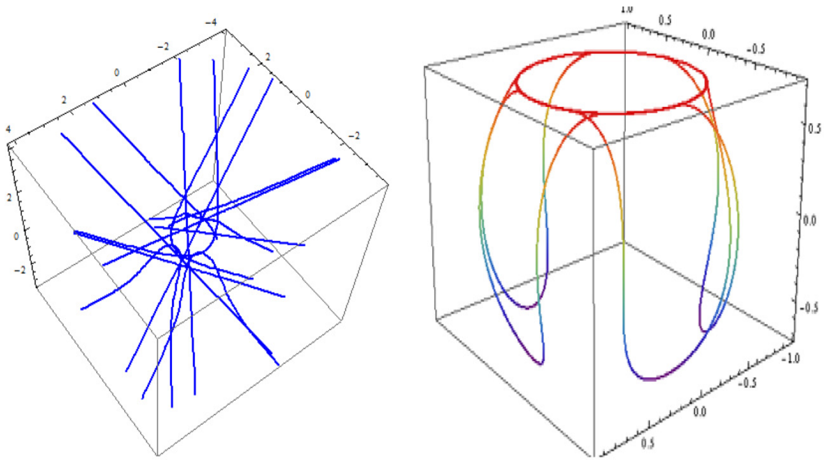

Figure 2. The modified centrode $\overline{D}$ of α and the spherical curve $Y(s)$.

Example 3.2. *We draw the picture of the curve*

$$\beta(s) = \left(3s - s^3, 3s^2, 3s + s^3\right)$$

in figure 3. Then we illustrate its modified centrode $\overline{D}(s)$ and the spherical curve $Y(s)$ regarded $\overline{D}(s)$ in figure 4.

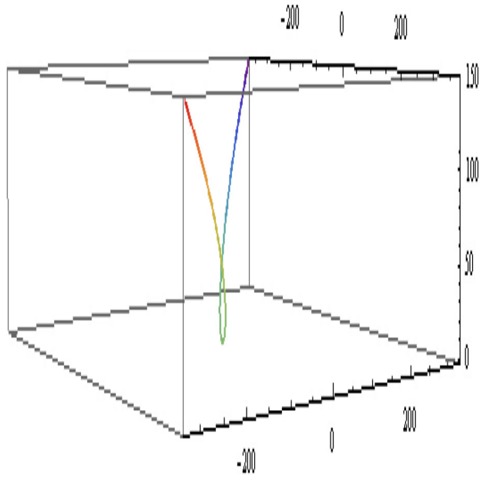

Figure 3. The curve $\beta(s)$.

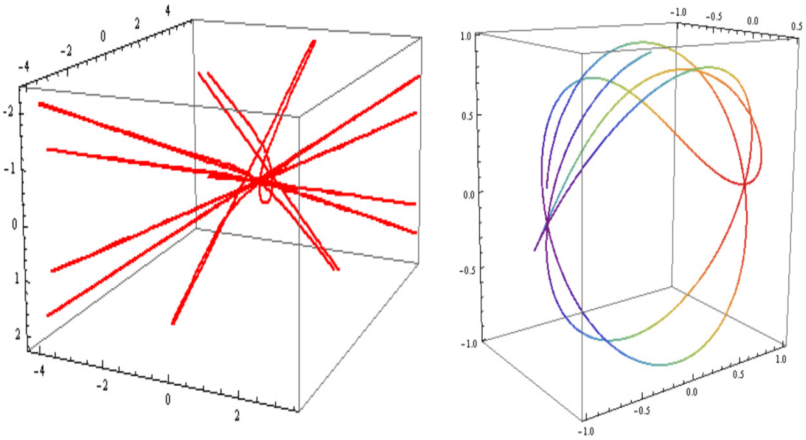

Figure 4. The modified centrode $\overline{D}$ of β and the spherical curve $Y(s)$.

4. Notes on rectifying developable surfaces

In differential geometry, a developable surface has zero Gaussian curvature. In three dimensions all developable surfaces are ruled surfaces. A ruled surface in $\mathbb{E}^3$ is the map $F_{(\alpha,\delta)} : I \times \mathbb{R} \to \mathbb{E}^3$ defined by $F_{(\alpha,\delta)}(s,u) = \alpha(s) + u\delta(s)$ where α is a base curve and δ is the director curve.

Let α be a unit speed curve with non-zero curvature $\varkappa$. Izumiya and Takeuchi [11] defined the following developables of the curve α.

Definition 4.1. *Let α be a unit speed curve with $\varkappa \neq 0$ in Euclidean space $\mathbb{E}^3$ with Frenet apparatus $\{T, N, B, \varkappa, \tau\}$. A ruled surface $F_{(\alpha,\widetilde{D})}(s,u) = \alpha(s) + u\,\widetilde{D}(s)$ is called the rectifying developable of α, $F_{(B,T)}(s,u) = B(s) + u\,T(s)$ is called the Darboux developable of α and $F_{(\widetilde{\widetilde{D}},N)}(s,u) = \widetilde{\widetilde{D}}(s) + uN(s)$ is called the tangential Darboux developable of α.*

In this study, firstly we defined extended rectifying developables of the curve α and then we characterize some special types of ruled surface obtained by choosing the base curve as one of the integral curves of Frenet vector fields and the director curve δ as the extended modified Darboux vector fields.

Definition 4.2. *Let α be a unit speed curve in Euclidean space $\mathbb{E}^3$ with any orthonormal frame apparatus $\{T, N_2, N_3, \varkappa_1, \varkappa_3\}$. We define the following extended rectifying developables of the curve α;*

- **i:** Considering $\dfrac{\varkappa_3}{\varkappa_1}$ is nonconstant, we define $\widetilde{F}_{(\widetilde{\alpha},\widetilde{\mathbf{D}})}(s,u) = \widetilde{\alpha}(s) + u\,\widetilde{\mathbf{D}}(s)$ where $\widetilde{\alpha}(s) = \int N_1(s)ds$.
- **ii:** Considering $\dfrac{\varkappa_1}{\varkappa_3}$ is nonconstant, we define $\overline{F}_{(\overline{\alpha},\overline{\mathbf{D}})}(s,u) = \overline{\alpha}(s) + u\,\overline{\mathbf{D}}(s)$ where $\overline{\alpha}(s) = \int N_3(s)ds$.

Theorem 4.1. *Let α be a unit speed curve in Euclidean space $\mathbb{E}^3$ with orthonormal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$.*

- **i:** Let $\widetilde{\alpha}(s) = \int N_1(s)ds$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with nonconstant ratio $\dfrac{\varkappa_3}{\varkappa_1}$. Then the curve $\widetilde{\alpha}(s)$ is a conical geodesic if and only if the rectifying developable $\widetilde{F}_{(\widetilde{\alpha},\widetilde{\mathbf{D}})}$ is a conical surface.

ii: Let $\overline{\alpha}(s) = \int N_3(s)ds$ be a unit speed curve in Euclidean space $\mathbb{E}^3$ with nonconstant ratio $\frac{\varkappa_1}{\varkappa_3}$. Then the curve $\overline{\alpha}(s)$ is a conical geodesic if and only if the rectifying developable $\overline{F}_{(\overline{\alpha},\overline{\mathbf{D}})}(s,u)$ is a conical surface.

Proof. Assume that the curve $\widetilde{\alpha}(s)$ is a conical geodesic. Then the singular locus of $\widetilde{F}_{(\widetilde{\alpha},\widetilde{D})}$ is given by

$$f(s) = \widetilde{\alpha}(s) - \frac{1}{\left(\frac{\varkappa_3}{\varkappa_1}\right)'(s)}\widetilde{D}(s).$$

Differentiating the last equation, we get

$$f'(s) = \left(\frac{\varkappa_3}{\varkappa_1}\right)''(s)\frac{1}{\left(\frac{\varkappa_3}{\varkappa_1}\right)'^2(s)}\widetilde{D}(s).$$

Since $\widetilde{\alpha}(s)$ is a conical geodesic we have $\frac{\varkappa_3}{\varkappa_1} = \lambda s + \mu$, $\lambda \in \mathbb{R}-\{0\}$ and $\mu \in \mathbb{R}$. So, $\left(\frac{\varkappa_3}{\varkappa_1}\right)'' = 0$ and therefore, $f'(s) = 0$. Consequently, we can easily see that the rectifying developable $\widetilde{F}_{(\widetilde{\alpha},\widetilde{\mathbf{D}})}$ is a conical surface.

Conversely, if the rectifying developable $\widetilde{F}_{(\widetilde{\alpha},\widetilde{\mathbf{D}})}$ is a conical surface then we can easily see that the curve $\widetilde{\alpha}(s)$ is a conical geodesic.

Similarly, we can easily proved this fact for the curve $\overline{\alpha}$. These complete the proof. □

Corollary 4.1. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed* ***nonhelical curve*** *in Euclidean space $\mathbb{E}^3$ with Frenet frame apparatus $\{T, N, B, \varkappa, \tau\}$. Then, the rectifying developable $\widetilde{F}_{(\alpha,\widetilde{D})}(s,u) = \alpha(s) + u\,\widetilde{D}(s)$ is a conical surface if and only if one of the curve α is a conical geodesic curve.*

Similarly, the rectifying developable $\widetilde{F}_{(\alpha,\overline{D})}(s,u) = \alpha(s) + u\,\overline{D}(s)$ is a conical surface if and only if the curve α is a conical geodesic curve.

Corollary 4.2. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed* ***nonslant curve*** *in Euclidean space $\mathbb{E}^3$ with the frame apparatus $\left\{N, C = \frac{N'}{\|N'\|}, W = N \times C, f, g\right\}$ defined by Uzunoğlu et al* [15]. *Then, the rectifying developable $\widetilde{F}_{(\widetilde{\alpha},\widetilde{\mathbb{D}})}(s,u) = \widetilde{\alpha}(s) + u\,\widetilde{\mathbb{D}}(s)$ is a conical surface if and only if one of the curve $\widetilde{\alpha} = \int N(s)ds$ is a conical geodesic curve with non-constant ratio $\frac{g}{f}$ and $f \neq 0$.*

Similarly, the rectifying developable $\overline{F}_{(\overline{\alpha},\overline{\mathbb{D}})}(s,u) = \overline{\alpha}(s) + u\,\overline{\mathbb{D}}(s)$ is a conical surface if and only if one of the curve $\overline{\alpha} = \int W(s)ds$ is a conical geodesic curve with non-constant ratio $\frac{f}{g}$ and $g \neq 0$.

Theorem 4.2. *Let α be a unit speed curve in Euclidean space $\mathbb{E}^3$ with orthonormal frame apparatus $\{N_1, N_2, N_3, \varkappa_1, \varkappa_3\}$. Then, the tangential Darboux developable $\widetilde{\widetilde{F}}_{(\alpha,\widetilde{\widetilde{\mathbf{D}}})}(s,u) = \widetilde{\widetilde{\mathbf{D}}}(s) + uN_2(s)$ is a conical surface if and only if the curve α is a N_2-slant curve, i.e., its vector field N_2 makes a constant angle with the constant vector field X.*

Proof. The singular locus of the the tangential Darboux developable $\widetilde{\widetilde{F}}_{(\alpha,\widetilde{\widetilde{\mathbf{D}}})}(s,u)$ is given by

$$\phi(s) = \widetilde{\widetilde{\mathbf{D}}}(s) + \sigma(s)N_2(s)$$

where $\sigma(s) = \dfrac{\varkappa_1^2\left(\dfrac{\varkappa_3}{\varkappa_1}\right)'}{\left(\varkappa_1^2+\varkappa_3^2\right)^{3/2}}$. Therefore, $\widetilde{\widetilde{F}}_{(\alpha,\widetilde{\widetilde{\mathbf{D}}})}(s,u)$ is a conical surface iff $\phi'(s) = 0$. Since $\widetilde{\widetilde{\mathbf{D}}}'(s) = -\sigma(s)N_2'(s)$ we get $\phi'(s) = -\sigma'(s)N_2(s)$. Hence, $\sigma(s)$ is a constant function, that is, α is a N_2-slant curve. □

Corollary 4.3. *Let $\alpha : I \subset \mathbb{R} \to \mathbb{E}^3$ be a unit speed* ***nonslant curve*** *in Euclidean space $\mathbb{E}^3$ with the frame apparatus $\left\{N, C = \dfrac{N'}{\|N'\|}, W = N \times C, f, g\right\}$ defined by Uzunoğlu et al* [15]*. Then, the tangential Darboux developable $\widetilde{\widetilde{F}}_{(\widetilde{\widetilde{\mathbb{D}}},C)}(s,u) = \widetilde{\widetilde{\mathbb{D}}}(s) + uC(s)$ is a conical surface if and only if the curve α is a $C-$slant curve.*

5. Conclusions

In this paper, we have obtained rectifying curves by using the modified Darboux vector fields corresponding to any orthonormal frame along a curve. For example, the orthonormal frame can be considered as some well-known frames such as the Frenet frame, the Darboux frame, the Bishop frame or the $\{N, C, W\}$ frame. In particular, choosing the Frenet frame gives some results obtained in [9].

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